AIAA JOURNAL
Vol. 37, No. 1, January 1999

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Comment on the Aeroacoustic Formulation of Hardin and Pope

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I. Introduction

IN Refs. 1 and 2, Hardin and Pope proposed a nonlinear two-step procedure for computational aeroacoustics that is suitable for both generation and propagation. In this approach, the viscous flow is obtained from the incompressible Navier–Stokes equations, and a correction to the constant hydrodynamic density is defined. Once this is defined, the acoustic radiation is obtained from the numerical solution of a system of perturbed, compressible, and inviscid equations. Changing some of the governing variables, however, we find inconsistencies in the formulation. We describe in detail the inconsistency and propose a new formulation that solves the problem.

II. Formulation of the Problem

Consider a viscous compressible flow around a body. At infinity, the flow moves with a velocity $U_\infty(t)$ in a reference system $(ox_1x_2x_3)$. Let (u_1, u_2, u_3) be the velocity, and ρ, p, T, and S be density, static pressure, temperature, and entropy per unit mass, respectively. Then the motion is governed by the compressible Navier–Stokes equations[3]

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_i)}{\partial x_i} = 0 \tag{1}$$

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j + p_{ij})}{\partial x_j} = 0 \tag{2}$$

$$p = p(\rho, S) \tag{3}$$

$$T\frac{\mathrm{D}S}{\mathrm{D}t} = c_p\frac{\mathrm{D}T}{\mathrm{D}t} - \frac{\beta T}{\rho}\frac{\mathrm{D}p}{\mathrm{D}t} = \chi + \frac{1}{\rho}\frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right) \tag{4}$$

where c_p, β, and k are the specific heat capacity at constant pressure, the coefficient of thermal expansion, and the coefficient of thermal conductivity, respectively; χ is the viscous dissipation; and

$$p_{ij} = p\delta_{ij} + \mu\left[-\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} + \frac{2}{3}\left(\frac{\partial u_k}{\partial x_k}\right)\delta_{ij}\right]$$

where μ is the dynamic viscosity.

III. Incompressible Solution

The incompressible solution can be obtained by solving the incompressible Navier–Stokes equations

$$\frac{\partial U_i}{\partial t} + \frac{\partial(U_iU_j)}{\partial x_j} = -\frac{1}{\rho_0}\frac{\partial P}{\partial x_i} + \nu\frac{\partial^2 U_i}{\partial x_j \partial x_j} \tag{5}$$

$$\frac{\partial U_i}{\partial x_i} = 0 \tag{6}$$

where P, ρ_0, and U_i are the incompressible pressure, the ambient density, and the velocity components, respectively.

The pressure change from the ambient pressure p_0 is

$$\mathrm{d}p = P - p_0$$

Equation (3) implies that

$$\begin{aligned}\mathrm{d}p &= \left(\frac{\partial p}{\partial \rho}\right)_S \mathrm{d}\rho + \left(\frac{\partial p}{\partial S}\right)_\rho \mathrm{d}S \\ &= c^2\,\mathrm{d}\rho + \left(\frac{\partial p}{\partial S}\right)_\rho \mathrm{d}S\end{aligned} \tag{7}$$

where the speed of sound is defined by $c = \sqrt{(\partial p/\partial \rho)_S}$. Equation (7) is normally solved along with Eq. (4). In Ref. 3, Batchelor notes that the effects of viscosity and heat conduction are normally to modify the pressure distribution rather than to control the magnitude of the pressure variation, i.e., these effects are slow on an acoustic timescale. Introducing the time-averaged incompressible pressure distribution $\bar{P}$

$$\bar{P}(\boldsymbol{x}) = \lim_{T\to\infty}\frac{1}{T}\int_0^T P(\boldsymbol{x}, t)\,\mathrm{d}t$$

the pressure fluctuation in time $p - \bar{P}$ can be assumed isentropic, and the entropy changes only for the averaged pressure $\bar{P}$. This leads to making a decomposition of the pressure as

$$p(\rho, S) = p^*(\rho) + \bar{P}(S)$$

such that only the averaged incompressible pressure involves losses, whereas the pressure fluctuation is assumed isentropic. Taking the derivative with respect to time, we get

$$\begin{aligned}\frac{\partial p}{\partial t} &= \frac{\partial p^*}{\partial t} \\ &= \frac{\mathrm{d}p^*}{\mathrm{d}\rho}\frac{\partial \rho}{\partial t} = \left(\frac{\partial p}{\partial \rho}\right)_S \frac{\partial \rho}{\partial t} = c^2\frac{\partial \rho}{\partial t}\end{aligned} \tag{8}$$

implying that the energy equation (4) is not needed for describing the acoustics.

IV. Acoustic Solution

We now consider the compressible Navier–Stokes equations. First we discuss the formulation of Hardin and Pope.[1] Next, a new formulation, which can be applied both for isentropic flows and non-isentropic flows, will be derived.

Received April 30, 1998; revision received Sept. 12, 1998; accepted for publication Sept. 30, 1998.

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A. Formulation of Hardin and Pope

In the model of Hardin and Pope,[1] a hydrodynamic density correction ρ_1 to the ambient density ρ_0 is introduced as

$$\rho_1(\boldsymbol{x}, t) = \frac{P(\boldsymbol{x}, t) - \bar{P}(\boldsymbol{x})}{c_0^2} \tag{9}$$

where c_0 is the ambient speed of sound, P is the incompressible pressure, and $\bar{P}$ is the time-averaged incompressible pressure distribution.

A set of perturbation equations is derived by dividing the variables in an incompressible part and a compressible perturbation part, as

$$u_i = U_i + u_i' \tag{10}$$

$$p = P + p' \tag{11}$$

$$\rho = \rho_0 + \rho_1 + \rho' \tag{12}$$

where u_i' and p' are the fluctuations of the velocity components and pressure about their incompressible counterparts and ρ' is the fluctuation of the density about the corrected incompressible density $\rho_0 + \rho_1$.

In the formulation of Hardin and Pope,[1] it is assumed that $p' = p'(\rho) = p(\rho, S) - P(\rho_0, S)$, implying that

$$\frac{\partial p'}{\partial \rho} = \left(\frac{\partial p}{\partial \rho}\right)_S = c^2 \tag{13}$$

and

$$\frac{\partial p'}{\partial t} = \frac{\partial p'}{\partial \rho}\frac{\partial \rho}{\partial t} = c^2\frac{\partial \rho}{\partial t} \tag{14}$$

Inserting Eqs. (10–12) into Eqs. (1), (2), and (14) and neglecting the effect of viscosity on the fluctuations, a set of nonlinear equations for the fluctuations is obtained as

$$\frac{\partial \rho'}{\partial t} + \frac{\partial f_i}{\partial x_i} = -\frac{\partial \rho_1}{\partial t} - U_i\frac{\partial \rho_1}{\partial x_i} \tag{15}$$

$$\frac{\partial f_i}{\partial t} + \frac{\partial}{\partial x_j}[f_i(U_j + u_j') + (\rho_0 + \rho_1)U_i u_j' + p'\delta_{ij}] = -\frac{(\rho_1 U_i)}{\partial t} - U_j\frac{\partial(\rho_1 U_i)}{\partial x_j} \tag{16}$$

$$\frac{\partial p'}{\partial t} - c^2\frac{\partial \rho'}{\partial t} = c^2\frac{\partial \rho_1}{\partial t} \tag{17}$$

where $f_i = (\rho_0 + \rho_1)u_i' + \rho'(U_i + u_i')$ and c^2 is calculated by

$$c^2 = \frac{\gamma p}{\rho} = \frac{\gamma(P + p')}{\rho_0 + \rho_1 + \rho'}$$

with γ the ratio of specific heats.

These equations constitute a closed set for the acoustic perturbation variables ρ', p', and u_i' with the right-hand-side expressions given from the incompressible solution. A closer look at these equations, however, shows that they contain no source terms. This can be seen if one introduces the variables $\tilde{\rho} = \rho_1 + \rho'$ and $\tilde{f}_i = \rho u_i' + \tilde{\rho} U_i$ instead of ρ' and f_i. Rearranging Eqs. (15–17), we get

$$\frac{\partial \tilde{\rho}}{\partial t} + \frac{\partial \tilde{f}_i}{\partial x_i} = 0 \tag{18}$$

$$\frac{\partial \tilde{f}_i}{\partial t} + \frac{\partial}{\partial x_j}\left[\tilde{f}_i(U_j + u_j') + \rho_0 U_i u_j' + p'\delta_{ij}\right] = 0 \tag{19}$$

$$\frac{\partial p'}{\partial t} - c^2\frac{\partial \tilde{\rho}}{\partial t} = 0 \tag{20}$$

It is seen that the system contains no source terms; hence, $p' = u_i' = \tilde{\rho} = 0$ is the solution to the acoustic system with initial and boundary conditions

$$p'|_{t=t_0} = 0, \qquad u_i'|_{t=t_0} = 0, \qquad \tilde{\rho}|_{t=t_0} = 0, \qquad u_n'|_{\Gamma} = 0$$

implying that solving Eqs. (15–17) results in the trivial solution $p' = u_i' = 0$ and $\rho' = -\rho_1$. On the other hand, Eqs. (15–17) have been solved for various cases giving nontrivial solutions.[1,2,4]

Some of these cases involved isentropic flows, and here Eq. (17) was replaced by the isentropic relation[1,4]

$$p/p_0 = (\rho/\rho_0)^\gamma$$

resulting in the algebraic equation

$$p' - p_0\left[\frac{(\rho_0 + \rho_1 + \rho')}{\rho_0}\right]^\gamma = -P \tag{21}$$

where the incompressible pressure P acts explicitly as a source term. Differentiating Eq. (21) with respect to t, we get

$$\frac{\partial p'}{\partial t} - c^2\frac{\partial \rho'}{\partial t} = c^2\frac{\partial \rho_1}{\partial t} - \frac{\partial P}{\partial t}$$

This is not consistent with Eq. (17) in which the term $-\partial P/\partial t$ is missing.

The inconsistency of Eqs. (15–17) is to be found in the assumption $P = P(\rho_0, S)$. From Eq. (10) we have that

$$P = c_0^2\rho_1 + \bar{P}$$

thus, $P = P(\rho_1)$ and the relationship in Eq. (13) does not hold.

B. New Formulation

From Eqs. (18–20) it was demonstrated that the introduction of a hydrodynamic density correction did not contain any new information. There is, therefore, no need to introduce this intermediate step, and instead we use the following decomposition for the compressible solution:

$$u_i = U_i + u_i' \tag{22}$$

$$p = P + p' \tag{23}$$

$$\rho = \rho_0 + \rho' \tag{24}$$

where ρ' is the fluctuating density about ρ_0.

Inserting Eqs. (22–24) into Eqs. (1), (2), and (8) and neglecting the viscous terms, we get the formulation

$$\frac{\partial \rho'}{\partial t} + \frac{\partial f_i}{\partial x_i} = 0 \tag{25}$$

$$\frac{\partial f_i}{\partial t} + \frac{\partial}{\partial x_j}[f_i(U_j + u_j') + \rho_0 U_i u_j' + p'\delta_{ij}] = 0 \tag{26}$$

$$\frac{\partial p'}{\partial t} - c^2\frac{\partial \rho'}{\partial t} = -\frac{\partial P}{\partial t} \tag{27}$$

where $f_i = \rho u_i' + \rho' U_i$.

The formulation is accomplished by the approximation that, for a general airflow,

$$c^2 = \frac{\gamma p}{\rho} = \frac{\gamma(P + p')}{\rho_0 + \rho'}$$

where $\gamma = 1.4$ is the ratio of specific heats.

Using Eq. (25), Eq. (27) can be replaced by

$$\frac{\partial p'}{\partial t} + c^2\frac{\partial f_i}{\partial x_i} = -\frac{\partial P}{\partial t} \tag{28}$$

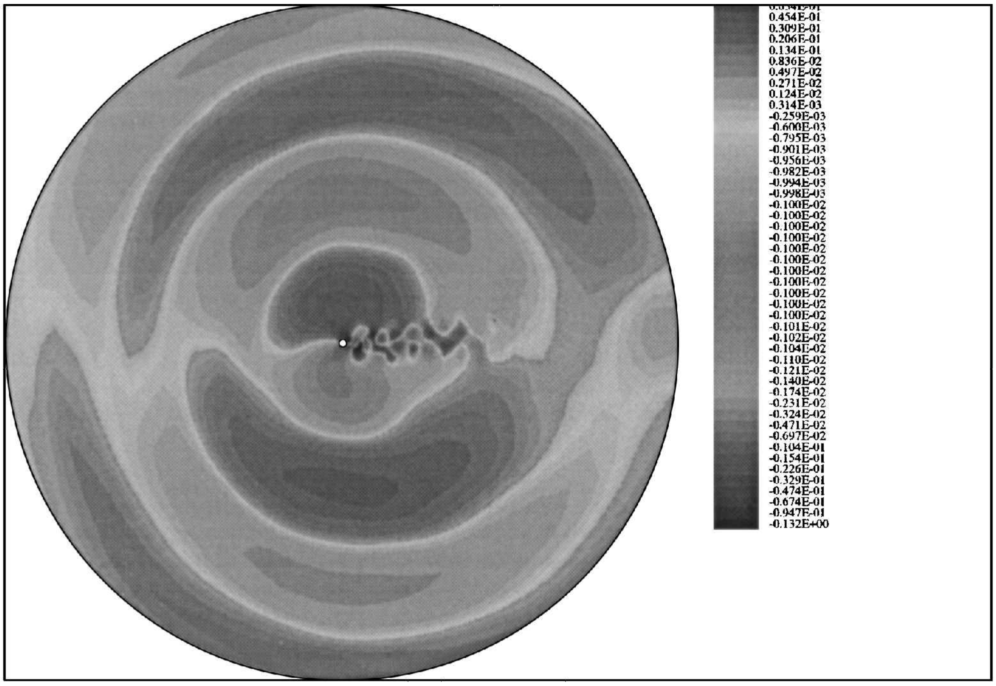

Fig. 1 Fluctuating pressure plot of flow past a circular cylinder at $Re = 200$ and $M = 0.2$.

Note that the only information coming from the incompressible solution is the instantaneous pressure and the acoustic calculation can be started at any time of the incompressible computation.

Proposition: For isentropic flow, the preceding formulation is equivalent to the formulation using the isentropic formula

$$p/p_0 = (\rho/\rho_0)^\gamma \tag{29}$$

Proof: The proof is two part: 1) implication and 2) inversion.

1) If the flow is supposed isentropic, one has $p = p(\rho)$, and Eq. (27) implies

$$\frac{\partial(p' + P)}{\partial t} = \frac{\partial p}{\partial t} = \frac{\mathrm{d}p}{\mathrm{d}\rho}\frac{\partial \rho}{\partial t} = c^2 \frac{\partial \rho}{\partial t}$$

Then one gets

$$\frac{\mathrm{d}p}{\mathrm{d}\rho} = c^2 = \frac{\gamma p}{\rho} \tag{30}$$

Integrating Eq. (30), we have $p/\rho^\gamma = \text{const}$. And then formula (29) is obtained.

2) Using Eq. (29), we get

$$\frac{\partial p}{\partial t} = \frac{\gamma p_0}{\rho_0}\left(\frac{\rho}{\rho_0}\right)^{\gamma - 1}\frac{\partial \rho}{\partial t}$$
$$= \frac{\gamma p}{\rho}\frac{\partial \rho}{\partial t}$$
$$= c^2 \frac{\partial \rho}{\partial t}$$

V. Results

In this section, the flow past a circular cylinder at Reynolds number 200 and Mach number 0.2 is considered as an illustration of the new approach. The detailed numerical scheme is referred to Ref. 5.

The incompressible solution is calculated in the cylindrical domain $[1, 95] \times [0, 2\pi]$ with a grid consisting of 121×161 grid points, stretched in the r direction. The acoustic solution is obtained in a domain $[1, 85] \times [0, 2\pi]$ with a grid consisting of 61×81 grid points.

To analyze the acoustic waves generated by the vortex shedding frequency, an unsteady (periodic) incompressible solution of the flow is obtained by introducing a small perturbation in the beginning of the computation. After a nondimensional time of $t = 50$, the incompressible periodic state is established, and the acoustic computation is started. In Fig. 1 the fluctuating pressure p' is plotted at a time instant $t = 600$. The waves are seen to propagate from the cylinder along the normal direction of the flow.

VI. Concluding Remarks

A new formulation for acoustic noise generation has been developed. The approach consists of two steps: an incompressible part and an acoustic part. The acoustic part can be started at any time during the incompressible computation. The model can be applied to both isentropic and nonisentropic flows. The computing costs are similar to what is typical for incompressible calculations.

Acknowledgments

This work was supported by the Danish Scientific Research Council under Grant 9700731 and EFP-98, the Research Programme for Renewable Energy under the Danish Energy Agency.

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